DAMPED VIBRATIONS OF HEREDITARY-ELASTIC SYSTEMS WITH WEAKLY SINGULAR KERNELS

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The use of hereditary kernels [1] with a weak singularity at the initial instant of time ($t = 0$) in solving dynamic problems of the theory of linear viscoelasticity reveals some interesting features of the behavior of the dynamic characteristics. This is because the corresponding relaxation (retardation) spectra have a singularity of the same type with respect to the relaxation time τ at $\tau = 0$, while the singularity parameter γ $(0 < \gamma \leq 1)$ characterizes the broadening of the spectrum. In Fourier space this singular point corresponds to an infinitely large frequency ($\omega = \infty$) [2], and therefore, in stationary dynamical problems, when the characteristics of the system are determined by the product $\omega\tau$, the singularity is expressed only through the parameter γ and appears in explicit form if τ or ω enters independently. This is clearly illustrated by the characteristics of an acoustic wave propagating in an hereditary-elastic medium [3].

The solution of nonstationary dynamical problems reveals other interesting singularities, investigated in [5] with reference to the example of a one-dimensional oscillator using Rabotnov's function [4] as relaxation kernel. It is worthwhile continuing the investigation of the damped vibration regime, especially in the presence of intense dissipative processes, which in the case of a delta spectrum lead to the appearance of aperiodic motions [6].

1. We will consider the question of freely damped vibrations in relation to the example of a single-mass system in motion as a result of an initial impulse. By virtue of the Boltzmann–Volterra hereditary elastic relations, the equation of motion can be written in two equivalent forms, either in terms of the relaxation kernel R(t) or in terms of the aftereffect kernel K(t)

$$\ddot{x} + \omega_\infty^2\left[x - \nu_\varepsilon \int\limits_{-\infty}^{t} R(t-t')x(t')\,dt'\right] = F\delta(t) \tag{1.1}$$

$$\ddot{x} + \omega_\infty^2 x + \nu_\sigma \int\limits_{-\infty}^{t} K(t-t')\ddot{x}(t')\,dt' = F\left[\delta(t) + \nu_\sigma K(t)\right]$$
$$\nu_\varepsilon = (\omega_\infty^2 - \omega_0^2)\,\omega_\infty^{-2}, \quad \nu_\sigma = (\omega_\infty^2 - \omega_0^2)\,\omega_0^{-2} \tag{1.2}$$

Here, x is a coordinate, F the pulse amplitude per unit mass, $\delta(t)$ the Dirac delta function, ω_∞ and ω_0 the frequencies of the elastic vibrations corresponding to the unrelaxed values of the elastic modulus; derivatives with respect to time are denoted by dots.

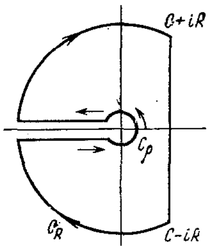

Fig. 1

The solution of Eqs. (1.1) and (1.2) in Laplace space is written in the form

$$x_*(p) = \frac{F}{p^2 + \omega_\infty^2\left[1 - \nu_\varepsilon R_*(p)\right]} = \frac{F\left[1 + \nu_\sigma K_*(p)\right]}{\omega_\infty^2 + p^2\left[1 + \nu_\sigma K_*(p)\right]} \tag{1.3}$$

Here, the asterisk subscript denotes the unilateral Laplace transform of the corresponding function.

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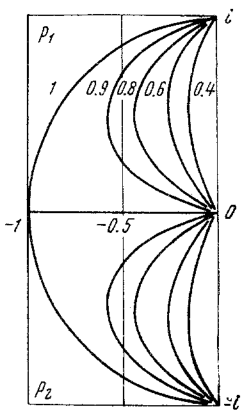

Fig. 2

The solution in the space of the inverse transforms is determined from the Mellin–Fourier inversion equation

$$x(t) = \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} x_*(p) e^{pt}\, dp \tag{1.4}$$

In order to evaluate integral (1.4), it is necessary to determine all the singular points of the complex function $x_*(p)$. The weakly singular kernels considered below have branch points ($p = 0$, $p = \infty$) and simple poles at values of p that make the denominators of Eqs. (1.3) vanish, i.e., that are roots of the equation

$$\begin{gathered} p^2 + \omega_\infty^2 [1 - \nu_\varepsilon R_*(p)] = 0 \\ \omega_\infty^2 + p^2 [1 + \nu_\sigma K_*(p)] = 0 \end{gathered} \tag{1.5}$$

The inversion theorem is applicable to multivalued functions with a branch point only for the first sheet of the Riemann surface, i.e., when $-\pi < \arg p < \pi$. Accordingly, the closed contour should be composed of the straight-line segment $[c - iR, c + iR]$, $c > 0$, the segments $-R < s < -\rho$ at the edges of the cut along the negative real semiaxis, and arcs of circles, one of which, C_ρ, $|p| = \rho$, closes the edge of the cut, while the other two, C_R, $|p| = R$, connect the edges of the cut with the vertical segment (Fig. 1). By virtue of Jordan's Lemma, as $R \to \infty$ the integrals along the curves C_R vanish. For weakly singular kernels the integral along C_ρ also tends to zero as $\rho \to 0$. Using the basic theorem of the theory of residues, we can write the solution of Eqs. (1.1) and (1.2) in the form

$$x(t) = \frac{1}{2\pi i} \int\limits_0^\infty [x_*(se^{-i\pi}) - x_*(se^{i\pi})]\, e^{-st}\, ds + \sum_k \operatorname{res}[x_*(p_k) e^{p_k t}] \tag{1.6}$$

Here the sum is taken over all the isolated singular points (poles).

2. As an example, we will consider the simplest case of weakly singular kernels – the Abel aftereffect kernel:

$$K(t) = [\Gamma(\gamma) \tau_\sigma^\gamma]^{-1} t^{\gamma-1}, \quad K_*(p) = (p\tau_\sigma)^{-\gamma}, \quad \tau_\sigma^\gamma E_0 = \tau_\varepsilon^\gamma E_\infty \tag{2.1}$$

The resolvent kernel of kernel (2.1) – the relaxation kernel – is given by the Rabotnov function:

$$\begin{gathered} R(t) = \tau_\varepsilon^{-\gamma} \partial_\gamma(-\nu, \tau, t) \\ \partial_\gamma(-\nu, \tau, t) = t^{\gamma-1} \sum_{n=0}^{\infty} \left(\frac{t}{\tau}\right)^{\gamma n} \frac{(-\nu)^n}{\Gamma[\gamma(n+1)]} \end{gathered} \tag{2.2}$$

Here, $\Gamma(\gamma)$ is the gamma function; γ is the singularity parameter (0, 1]; and τ_ε, τ_σ are the relaxation and retardation times, respectively. In Eq. (2.2) and in what follows, where the quantities τ, ν are written with out indices, it is assumed that $\tau = \tau_\varepsilon$ if $\nu = \nu_\varepsilon$, and $\tau = \tau_\sigma$ if $\nu = \nu_\sigma$.

Substituting Eqs. (2.1) and (2.2) in (1.3), we find

$$x_*(p) = F(p^\gamma + \varkappa) p^{-\gamma} (p^2 + \varkappa p^{2-\gamma} + \omega_\infty^2)^{-1}, \quad \varkappa = \nu\tau^{-\gamma} \tag{2.3}$$

In order to determine the poles of the function $x_*(p)$, it is necessary to find the roots of the equation

$$p^2 + \varkappa p^{2-\gamma} + \omega_\infty^2 = 0 \tag{2.4}$$

It is easy to see that Eq. (2.4) does not have real negative roots. In fact, setting $p = -y$, $y > 0$ in (2.4), we obtain an equation that contradicts the starting assumption.

In order to find the roots of Eq. (2.4), we set $p = re^{i\psi}$. Then, separating the real and imaginary parts, we obtain a system of two equations:

$$\begin{gathered} r^2 \cos 2\psi + \varkappa r^{2-\gamma} \cos(2-\gamma)\psi + \omega_\infty^2 = 0 \\ r^2 \sin 2\psi + \varkappa r^{2-\gamma} \sin(2-\gamma)\psi = 0 \end{gathered} \tag{2.5}$$

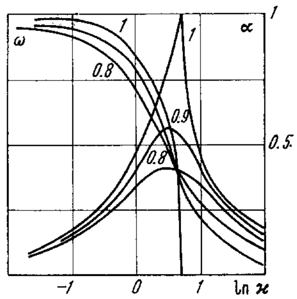

Fig. 3

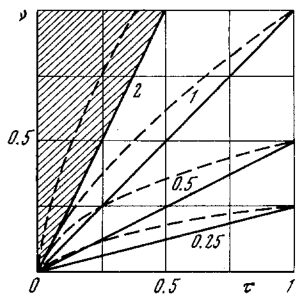

Fig. 4

Here, it has been assumed that in the elastic region $\omega_\infty = 1$. Obviously, system (2.5) does not have roots at any $|\psi| < 1/2\pi$. In order to calculate the roots of system of equations (2.5), we introduce the new variables $x_1 = r^2$ and $x_2 = \varkappa r^{2-\gamma}$. Then for each fixed angle $1/2\pi < \psi < \pi$ and given γ we obtain a system of two linear equations in two unknowns x_1 and x_2. After finding the values of x_1 and x_2, we find r and $\varkappa$, which, together with the selected ψ, determine the root of the characteristic equation. Upon substituting $-\psi$ for ψ we obtain the conjugate complex root.

Thus, in the plane with the cut-out negative real semiaxis $-\pi < \psi \leq \pi$, Eq. (2.4)has two conjugate complex roots $p_{1,2} = -\alpha \pm i_\omega$ at each fixed $\varkappa$. The behavior of these roots as functions of $\varkappa$ is shown in Fig. 2, where as parameter we have selected the quantity γ, whose values are indicated by the figure adjacent to the curves. At $\gamma = 1$ characteristic equation (2.4) corresponds, correct to a constant, with Maxwell's rheological model and the roots are given by the equation

$$p_{1,2} = -\tfrac{1}{2}\varkappa \pm (\tfrac{1}{4}\varkappa^2 - 1)^{1/2} \tag{2.6}$$

It follows from (2.6) that at $\varkappa \geq 2$ there are two real roots, one of which tends to zero as $\varkappa \to \infty$, while the other tends to infinity. As $\gamma \to 0$ the roots of Eq. (2.4) give the solution for undamped elastic vibrations (i.e., they are imaginary), and as $\varkappa$ varies from 0 to infinity, they vary from i_{ω_∞} to 0.

Knowing the behavior of the roots of the characteristic equation, we write the solution (1.6) in the form

$$x(t) = A_0(t) + A\exp(-\alpha t)\sin(\omega t - \varphi) \tag{2.7}$$

$$A = \frac{F}{r}\left[1 - \gamma\frac{\varkappa r^{-\gamma}(1 - \tfrac{1}{4}\gamma) + \cos\gamma\psi}{\varkappa r^{-\gamma} + \varkappa^{-1}r^{\gamma} + 2\cos\gamma\psi}\right]^{-1/2} \tag{2.8}$$

$$\operatorname{tg}\varphi = -\frac{2(\varkappa r^{-\gamma} + \varkappa^{-1}r^{\gamma} + 2\cos\gamma\psi)\cos\psi - \gamma[\varkappa r^{-\gamma}\cos\psi + \cos(1-\gamma)\psi]}{2(\varkappa r^{-\gamma} + \varkappa^{-1}r^{\gamma} + 2\cos\gamma\psi)\sin\psi - \gamma[\varkappa r^{-\gamma}\sin\psi + \sin(1-\gamma)\psi]}$$

$$r^2 = \omega^2 + \alpha^2, \quad \operatorname{tg}\psi = -\omega\alpha^{-1}$$

$$A_0(t) = \int_0^\infty \tau^{-1}B(\tau)e^{-t/\tau}\,d\tau$$

The quantity $A_0(t)$ describing the elastic aftereffect may be regarded as the Laplace transform of the spectral function

$$B(\tau) = \frac{\sin\pi\gamma}{\pi}\frac{F\omega_\infty^2\tau^3(1+\omega_\infty^2\tau^2)^{-1}}{(\tau^\gamma\varkappa)^{-1}(1+\omega_\infty^2\tau^2) + \tau^\gamma\varkappa(1+\omega_\infty^2\tau^2)^{-1} + 2\cos\pi\gamma} \tag{2.9}$$

which gives the distribution of the relaxation parameters of the dynamical system.

In the quasistatic case distribution function (2.9) goes over into the Abel kernel retardation-time distribution function

$$B^{(\zeta)}(\tau)=\frac{F\nu_\sigma}{\omega_\infty^2}\frac{\sin\pi\gamma}{\pi\tau_\sigma^{\gamma}}\tau^{\gamma-1} \tag{2.10}$$

Expressions (2.7) and (2.8) define harmonic vibrations with natural frequency ω and damping coefficient α. The behavior of these quantities as a function of $\ln \varkappa$ is illustrated in Fig. 3. It is clear from Fig. 3 that α passes through a maximum, while ω decreases monotonically from 1 to 0 with increase in $\ln \varkappa$. The logarithmic decrement $\Delta = 2\pi\alpha\omega^{-1}$ is constant along the lines $\varkappa$ = const. In Fig. 4, we have plotted in $\nu_\varepsilon - \tau_\varepsilon$ coordinates the isodecremental lines for $\gamma = 1$ (solid lines) and $\gamma = 0.5$ (dashed lines). The values of $\varkappa$ are indicated by the figures adjacent to the curves. The shaded region above the straight line $\varkappa = 2$ ($\Delta = \infty$) corresponds to the region of aperiodicity, while the region beneath this straight line corresponds to the region of damped vibrations. If the elastic modulus relaxes completely, we obtain a quasi-Maxwellian model, which at $\gamma = 1$ goes over into the usual Maxwell model with the boundary of the region of aperiodicity at $\tau = 1/2$. For fractional γ, there is no region of aperiodicity.

It should be noted that as $\gamma \to 1$ solution (2.7) goes continuously over into the solution corresponding to the region of vibration of a Maxwell material. However, in this case it is not possible to obtain a solution corresponding to the region of aperiodicity.

Thus, we have been able to trace the effect of the parameter γ, characterizing the "smearing" of the corresponding spectrum, on the dynamic characteristics of the system: logarithmic decrement, natural frequency, and damping coefficient. Moreover, it is possible to establish the nature of the vibratory process, whose principle characteristic consists in the impossibility of the damped vibrations going over into the aperiodic mode.

It is known that if the intensity of the dissipative process is sufficiently large actual vibratory systems may exhibit aperiodic behavior. In order to describe this fact, it is necessary to use other distribution functions. Certain information relating to this question may be found, for example, in [7, 8].

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